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THE STABILITY OF NON-CONSERVATIVE SYSTEMS WITH SINGULAR MATRICES OF DISSIPATIVE FORCES†

V. N. KOSHLYAKOV and V. L. MAKAROV

Kiev

e-mail: institute@imath.kiev.ua

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Earlier results [1–4] are developed in application to a certain special class of non-conservative mechanical systems in which the matrices of dissipative and non-conservative forces are singular. For this class of systems, necessary and sufficient conditions are formulated for reducing the initial matrix equation to a form that admits of direct application of the Kelvin–Chetayev theorems. An example is presented.

A general investigation of the qualitative properties of non-conservative mechanical systems presents a rather complicated problem because of the unusual and not easily predicted influence of non-conservative positional forces on the stability of the system. In some cases, such forces actually promote expansion of the stable domain. Even a slight change in the system parameters, however, may mean that non-conservative forces will destroy stability.

A technique has been proposed [1–4] to investigate mechanical systems with non-conservative forces, based on the use of the Lyapunov matrix to transform the initial equation. The transformation, which does not usually affect the stability properties of the linear part of the initial equation, is constructed in such a way that the transformed equation does not involve non-conservative positional structures at all. In the context of the technique it was assumed [1] that the matrices D, G, Π and P (see Eq. (1.1) below) might be variable.

As is well known, asymptotic stability in linear systems that contain non-conservative positional forces cannot be guaranteed unless dissipative forces are taken into account. It is therefore significant that the technique considered in the papers cited above is applicable only on condition that the matrices of the dissipative and non-conservative positional forces are non-singular.

Another approach to a stability analysis of non-conservative systems has been considered, using Lyapunov's direct method without the above-mentioned structural transformation of the initial system; but then, too, the matrix of dissipative forces was also assumed to be non-singular [5].

Some recent publications have been devoted to the investigation of non-conservative systems in which the matrix of dissipative forces is singular. An example is a study of the stability of a body suspended on a string [6]. The matrix of dissipative forces in the equations of perturbed motion of such a system is singular, though the dissipation is nevertheless complete.

On the basis of the technique developed in [1–4], a certain class of systems in which the matrices of dissipative and non-conservative forces are also singular has been considered.‡ That study assumes that the matrices of dissipative and positional non-conservative forces, appearing as blocks in the structure of the initial matrices, satisfy a certain linear relation with a scalar constant.

Below we will consider a more general class of mechanical systems with singular matrices of dissipative and non-conservative positional forces, which does not require satisfaction of the above-mentioned linear

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relation. Rigorous conditions, on the basis of which stability can be investigated by direct application of the Kelvin–Chetayev theorems, will be obtained.

1. INITIAL EQUATIONS

Consider the matrix equation

$$J\ddot{x} + (D + HG)\dot{x} + (\Pi + P)x = 0 \tag{1.1}$$

where $x = \mathrm{col}(x_1, \dots, x_{2m})$ is an unknown vector; $J = J^T$, $D = D^T$, $G = -G^T$, $\Pi = \Pi^T$, $P = -P^T$ are given constant $2m \times 2m$ matrices, and H is a large positive scalar parameter, on which, generally speaking, the matrix Π may depend.

The matrix J is assumed to be positive-definite, and the matrix D, unlike the conditions assumed previously in [1–4], is assumed to be positive-semidefinite.

Equation (1.1) describes the behaviour of a large number of mechanical systems subject to the action of dissipative, gyroscopic, potential, and properly non-conservative positional forces. In systems containing gyroscopes, J must be understood as the matrix of the total moments of inertia about the appropriate axes.

Let us assume that $\mathrm{rank}D = m$. Then, using elementary operations, we can express matrix equation (1.1) in such a way that the matrix D has the form [7]

$$D = \begin{Vmatrix} D_{11} & 0 \\ 0 & 0 \end{Vmatrix} \tag{1.2}$$

We change to a new vector $\xi(t)$, putting

$$x(t) = L_1(t)\xi(t) \tag{1.3}$$

where the matrix $M_1(t)$ is determined from the conditions

$$D\dot{L}_1(t) = -PL_1(t); \quad P = \|P_{\alpha\beta}\|_{\alpha,\beta = 1,2}, \quad t > 0; \quad L_1(0) = \mathrm{diag}(E, E) \tag{1.4}$$

where E is the $m \times m$ identity matrix.

Conditions (1.4) will be satisfied if

$$P_{12} = P_{22} = 0, \quad L_1(t) = \mathrm{diag}[L(t), L(t)] \tag{1.5}$$

where the $m \times m$ matrix $L(t)$ is determined from the condition

$$\dot{L}(t) = -D_{11}^{-1}P_{11}L(t), \quad t > 0, \quad L(0) = E \tag{1.6}$$

Put

$$\begin{aligned} V_1 &= 2J\mathrm{diag}[A, A] + D + HG \\ W_1 &= J\mathrm{diag}[A^2, A^2] + HG\mathrm{diag}[A, A] + \Pi, \quad A = -D_{11}^{-1}P_{11} \end{aligned} \tag{1.7}$$

Then, using substitutions (1.3) and (1.6), we can reduce Eq. (1.1) to the form

$$\ddot{\xi}(t) + L_1^{-1}(t)J^{-1}V_1L_1(t)\dot{\xi}(t) + L_1^{-1}(t)J^{-1}W_1L_1(t)\xi(t) = 0 \tag{1.8}$$

where the matrix $L_1(t)$ is assumed to be non-singular. If the commutativity conditions

$$J^{-1}V_1L_1(t) = L_1(t)J^{-1}V_1, \quad J^{-1}W_1L_1(t) = L_1(t)J^{-1}W_1 \tag{1.9}$$

are satisfied, Eq. (1.8) reduces to the form

$$J\ddot{\xi}(t) + V_1\dot{\xi}(t) + W_1\xi(t) = 0, \tag{1.10}$$

where the matrices J, V_1 and W_1 are constant and defined by formulae (1.7).

If the matrix W_1 turns out to be symmetric and the matrix V_1 depends on the dissipative and gyroscopic forces, the Kelvin–Chetayev theorems are applicable to Eq. (1.10). It is therefore natural to formulate the problem of finding the necessary and sufficient conditions for Eqs (1.8) and (1.10) to be equivalent, in the sense that every solution of Eq. (1.8) is at the same time a solution of Eq. (1.10), and vice versa.

2. THE CONDITION FOR EQS (1.8) AND (1.10) TO BE EQUIVALENT

The solution of the Cauchy matrix problem (1.6) has the form

$$L(t) = \exp(At) = D_{11}^{-1/2}\exp(P'_{11}t)D_{11}^{1/2}, \quad P'_{11} = -D_{11}^{-1/2}P_{11}D_{11}^{-1/2} \tag{2.1}$$

provided that the matrix D_{11} is positive-definite, so that the matrices $D_{11}^{1/2}$ and $D_{11}^{-1/2}$ exist. If it is required that $\det P_{11} \neq 0$, then it can be shown, exactly as in [3], that the matrix $L(t)$, and together with it the matrix $L_1(t)$, are Lyapunov matrices. Hence transformation (1.3) does not affect the stability properties of Eq. (1.1).

Theorem. Let J, D and Π be arbitrary symmetric $2m \times 2m$ matrices such that J and D_{11} are positive-definite. Let P_{11} be an arbitrary non-singular skew-symmetric $m \times m$ matrix, where m is even. Then Eqs (1.8) and (1.10) will be equivalent for any $H > 0$ if and only if the following conditions hold

$$\begin{aligned} &J^{-1}G\dot{L}_1(0) = \dot{L}_1(0)J^{-1}G, \quad J^{-1}\Pi\dot{L}_1(0) = \dot{L}_1(0)J^{-1}\Pi, \\ &J^{-1}D\dot{L}_1(0) = \dot{L}_1(0)J^{-1}D, \quad \dot{L}_1(0) = \operatorname{diag}(A, A) \end{aligned} \tag{2.2}$$

Proof. We will first show that Eqs (1.8) and (1.10) will be equivalent if and only if conditions (1.9) hold. Indeed, suppose conditions (1.9) are satisfied. Then, since the matrix $L_1(t)$ $(t \geq 0)$ is non-singular, Eq. (1.8) is transformed into Eq. (1.10). Hence Eqs (1.8) and (1.10) are equivalent.

The converse also holds. Let Eqs (1.8) and (1.10) be equivalent in the sense indicated. Fix an arbitrary time $t_0 \geq 0$ and define

$$\xi(t_0) = e_k, \quad \dot{\xi}(t_0) = 0, \quad e_k = (\delta_{jk})_{j=1}^{2m}, \quad k = 1, 2, \ldots, 2m \tag{2.3}$$

(δ_{jk} is the Kronecker delta). Then it follows from Eq. (1.10) that

$$\ddot{\xi}(t_0) = -J^{-1}W_1e_k \tag{2.4}$$

Since the solution of Eq. (1.10) with initial data (2.3) is at the same time a solution of Eq. (1.8), we have

$$-JL_1(t_0)J^{-1}W_1e_k + W_1L_1(t_0)e_k = 0$$

or

$$J^{-1}W_1L_1(t_0)e_k = L_1(t_0)J^{-1}W_1e_k, \quad \forall t_0 \geq 0, \quad k = 1, 2, \ldots, 2m \tag{2.5}$$

which is the second condition of (1.9).

The first condition is proved similarly. Instead of initial data (2.3) we must take

$$\xi(t_0) = 0, \quad \dot{\xi}(t_0) = e_k, \quad k = 1, 2, \ldots, 2m \tag{2.6}$$

We then obtain the first of conditions (1.9). Repeating the arguments employed in [3], it can be shown that conditions (1.9) are equivalent to the conditions

$$J^{-1}V_1\dot{L}_1(0) = \dot{L}_1(0)J^{-1}V_1, \quad J^{-1}W_1\dot{L}_1(0) = \dot{L}_1(0)J^{-1}W_1 \tag{2.7}$$

Indeed, conditions (2.7) obviously follow from conditions (1.9). It then follows from (1.5) and (2.1) that the matrices $J^{-1}V_1$ and $J^{-1}W_1$ commute with the matrix $\operatorname{diag}[D_{11}^{-1}P_{11}, D_{11}^{-1}P_{11}]$, and hence also with the matrix $L(t)$, $\forall t \geq 0$.

At the same time, conditions (2.7) for any $H > 0$ hold if and only if the commutativity conditions (2.2) hold.

Corollary. The matrix W_1 will be symmetric for any $H > 0$ if and only if

$$G\dot{L}_1(0) = -\dot{L}_1(0)^T G, \quad J[\dot{L}_1(0)]^2 = [\dot{L}_1(0)]^2 J \tag{2.8}$$

The proof becomes obvious if the difference $W_1 - W_1^T$ is written in the form

$$W_1 - W_1^T = J[\dot{L}_1(0)]^2 - [\dot{L}_1(0)^T]^2 J + H\{G\dot{L}_1(0) + \dot{L}_1(0)^T G\} \tag{2.9}$$

By virtue of the foregoing discussion, to solve problem (2.4) it is not necessary to make a special determination of the pseudo-inverse of the matrix D^+[8], since it is obtained directly in explicit form

$$D^+ = \begin{Vmatrix} D_{11}^{-1} & 0 \\ 0 & 0 \end{Vmatrix} \tag{2.10}$$

It also follows from the proof of the theorem that the matrices G and Π may be variable, agreeing with earlier results [11].

On the assumption that the scalar parameter $H > 0$ is sufficiently large and the matrix of gyroscopic forces is non-singular, the so-called precession equations, whose matrix representation is obtained from Eq. (1.1) by neglecting the term $J\ddot{x}$ on the left of the equation, are very useful in the applied theory of gyroscopes. One obtains an equation of the form

$$(D + HG)\dot{u} + (\Pi + P)u = 0 \tag{2.11}$$

Of course, the legitimacy of replacing Eq. (1.1) by Eq. (2.11) requires justification. It has been established that one obstacle encountered in changing to the precession equations is the presence of non-conservative positional structures in the initial equations. In that case an asymptotically stable solution obtained by using Eq. (2.11) may turn out to be unstable in the exact equations, because of the divergence of rapid nutational oscillations. In that situation, if the dissipative forces that are always present in a real system are ignored, it is not possible to achieve stability by any domination of gyroscopic forces, i.e. by any increase in the scalar parameter H. This will be taken into consideration below when we consider a version of the four-gyroscope gyro horizon with control of the radial-correction type.

3. A FOUR-GYROSCOPE GYRO HORIZON

As an example of the application of the above theory, we will investigate a mathematical model of a four-gyroscope gyro horizon, which differs in its control structure from previously considered systems of this kind [9].

The system consists of a platform mounted in gimbals horizontally stabilized by means of four identical gyroscopes whose housings have vertical axes. The gyroscopes are coupled two by two by anti-parallelograms which allow each pair of gyroscopes to rotate in opposite directions through the same angle in the plane of the platform. Each pair of gyroscopes is attached by a spring to the inner frame of the suspension. It is assumed that the centre of mass of the system lies below its geometrical centre.

Unlike the systems considered in [9], two control systems, with operation of the radial-correction type, are provided. The controls apply two correcting torques: one torque, about the axis of the outer gimbal, is proportional to the angle through which one of the gyroscope pairs rotates about the vertical axes of their housings; the torque is applied about the axis of the housing of a gyroscope of the same pair, proportional to the angle through which the outer gimbal turns.

The equations of motion of such a system, assuming that it is mounted on a moving base and taking the Earth's rotation into account, have the form

$$\begin{aligned}
&J_1\ddot{\alpha} + b_1\dot{\alpha} + 2H\dot{\delta} + 2H\omega\gamma + s_1\delta + Pl\alpha = -Pl\upsilon\omega/g \\
&J_2\ddot{\delta} + b_2\dot{\delta} - 2H\dot{\alpha} + 2H\omega\beta + c\delta - s_2\alpha = 2HU\cos\varphi\sin\psi \\
&J_2\ddot{\gamma} + 2H\dot{\beta} + 2H\omega\alpha + c\gamma = -2H(U\cos\varphi\sin\psi + \upsilon/R) \\
&J_2\ddot{\beta} - 2H\dot{\gamma} + 2H\omega\delta + Pl\beta = 0
\end{aligned} \tag{3.1}$$

where α and β are the angles of deviation of the platform from the plane of the horizon, and γ and δ are the angles through which each gyroscope pair rotates about the vertical axes of their housings. The remaining notation is the same as in Eqs (3.1) of [3]. The following additional notation has been introduced: U is the angular velocity of the Earth's diurnal rotation, on the assumption that Earth is a sphere of radius R, φ is the (geocentric) latitude of the location and ψ is the angle measured in the clockwise direction from the direction to the north (the course of the base). The vertical component $U\sin\varphi$ of the angular velocity of the Earth's rotation is included in ω.

Allowance is made for the torques $b_1\dot{\alpha}$ and $b_2\dot{\delta}$ of small dissipative forces in the first two equations of system (3.1), which contain the non-conservative positional torques $s_1\delta$ and $s_2\alpha$. In the other two equations, which do not contain non-conservative positional structures, dissipative forces are ignored.

The homogeneous part of system (3.1) corresponds to the matrix equation (1.1) and the structures (1.2). Putting $\alpha = x_1$, $\delta = x_2$, $\gamma = x_3$, $\beta = x_4$ and introducing the vector $x = \operatorname{col}(x_1, x_2, x_3, x_4)$, we have, with reference to system (3.1)

$$J = \operatorname{diag}[J_1, J_2, J_2, J_3], \quad D = \operatorname{diag}[b_1, b_2, 0, 0], \quad D_{11} = \operatorname{diag}[b_1, b_2], \quad G = \operatorname{diag}[S, S]$$

$$P = s\operatorname{diag}[S, [0]], \quad P_{11} = sS, \quad S = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}$$

$$\Pi = \begin{Vmatrix} T_1 & 2H\omega E \\ 2H\omega E & T_2 \end{Vmatrix}, \quad T_1 = \begin{Vmatrix} Pl & m \\ m & c \end{Vmatrix}, \quad T_2 = \begin{Vmatrix} c & 0 \\ 0 & Pl \end{Vmatrix} \tag{3.2}$$

$$s = (s_1 + s_2)/2, \quad m = (s_1 - s_2)/2$$

For these data

$$\dot{L}_1(0) = \operatorname{diag}(A, A), \quad A \begin{Vmatrix} 0 & -b_1^{-1}s \\ b_2^{-1}s & 0 \end{Vmatrix} \tag{3.3}$$

The matrix A is determined in accordance with the expressions (1.5).

Conditions (2.2) impose certain restrictions on the choice of the parameters of the system. Thus, using the first condition and noting formulae (3.2), we obtain

$$b_1/b_2 = J_1/J_2 = J_2/J_3 \tag{3.4}$$

Taking the second condition of (2.2) into consideration, as well as the notation in (3.2), we have the further constraints

$$b_1 = b_2, \quad J_1 = J_2 = J_3, \quad s_1 = s_2, \quad c = Pl \tag{3.5}$$

The third of conditions (2.2) is satisfied in this situation and entails no further constraints.

Under conditions (3.4) and (3.5), in which we put $b_1 = b_2 = b, J_1 = J_2 = J_3 = J$, we obtain the following expression for the matrix W_1

$$W_1 = 2H\omega \begin{Vmatrix} \mu E & E \\ E & \mu E \end{Vmatrix}, \quad \mu = \frac{1}{2H\omega}\left[c + 2H\frac{s}{b} - J\left(\frac{s}{b}\right)^2\right] \tag{3.6}$$

where E is the 2×2 identity matrix.

Applying Sylvester's criterion to the symmetric matrix (3.6), we obtain a necessary and sufficient condition for it to be positive-definite

$$b^2c + 2Hb(s - b\omega) - Js^2 > 0 \tag{3.7}$$

which has the same structure as the second of conditions (3.8) in [3] if we put $J = 2A'$.

Note that on the assumption that the scalar parameter $H > 0$ is sufficiently large, we can apply the technique used in [4], comparing the norms of the appropriate matrices. Then, as shown in [4], one can derive conditions more general than (3.7), in particular, not requiring that $s_1 = s_2$ as in (3.5).

The matrix D of dissipative forces in Eqs (3.1) is singular. Hence the corresponding Rayleigh function $2\Phi = b_1\dot{\alpha}^2 + b_2\dot{\delta}^2$ will not be positive-definite for all velocities $\dot{\alpha}$, $\dot{\delta}$, $\dot{\gamma}$, $\dot{\beta}$. Nevertheless, the dissipation in Eqs (3.1) may be complete.

Looking at the homogeneous part of system (3.1), we see that if $\dot{\alpha} = \dot{\delta} = 0$, i.e. if α and δ are constant, the quantities β and γ will also be constant, $\dot{\beta} = \dot{\gamma} = 0$. This means that the dissipative forces vanish at equilibrium positions.

Let Δ denote the determinant of the positional forces in system (3.1). Computation gives

$$\Delta = (4H^2\omega^2 - cPl)^2 + cPls_1s_2 \tag{3.8}$$

If this determinant does not vanish, the dissipative forces will vanish only in the unperturbed motion, corresponding to equilibrium positions. This means that the dissipation is complete [6]. Then, if condition (3.7) is satisfied, so that the matrix W_1 is positive-definite, the addition of arbitrary gyroscopic forces and forces with complete dissipation gives the system the property of asymptotic stability.

If the determinant (3.8) turns out to be equal to zero, the system may not have the property of asymptotic stability. To demonstrate this, set $Pl = 0$, $c = 0$ in Eqs (3.1) for the case of a stationary base, when $\omega \equiv 0$. This case corresponds to the absence of a pendulum effect in the system, when its centre of gravity coincides with the geometric centre of the suspension, and also the absence of springs through which the gyroscopes are attached to the inner gimbal. Under these conditions, system (3.1) splits into two independent systems. The equations for the coordinates β and γ are then

$$J_2\ddot{\gamma} + 2H\dot{\beta} = M, \quad J_3\ddot{\beta} - 2H\dot{\gamma} = 0, \quad M = -2H(U\cos\varphi\sin\psi + v/R) \tag{3.9}$$

The homogeneous part of Eqs (3.9) corresponds to the initial equation (1.1) when only gyroscopic forces are present.

If the determinant of the matrix of gyroscopic forces does not vanish (as happens in this case), the system turns out to be stable in velocities and coordinates, but not asymptotically. Even by adding dissipative forces in system (3.9), one can ensure asymptotic stability only in velocities, but not in coordinates [10].

If there is a constant torque M on the right of Eqs (3.8), we obtain an undesirable drift in the system, increasing linearly with time. Indeed, the solution of system (3.9) for zero initial data in the coordinates and velocities has the form

$$\beta = \frac{Mt}{2H} + \frac{M}{2Hk}\sin kt, \quad \gamma = \frac{J_3M}{4H^2}(1 - \cos kt); \quad k = \frac{2H}{\sqrt{J_2J_3}} \tag{3.10}$$

The first of these expressions represents a systematic drift of the inner gimbal of the suspension, accompanied by undamped nutational oscillations at an angular frequency k, determined by the last formula of (3.10). The coordinate γ experiences nutational oscillations at the same frequency, about an equilibrium position $\gamma = \gamma^*$, where $\gamma^* = J_3M/(4H^2)$, which is displaced from zero and very small in magnitude.

The presence of a systematic drift of the inner gimbal also follows directly from Eqs (3.9), considered in the context of the precession theory, for which the inertial terms $J_2\ddot{\gamma}$ and $J_3\ddot{\beta}$ must be ignored in the equations specified.

Another result is obtained if, when $Pl = c = 0$, one assumes that $\omega \neq 0$. For these conditions, by formula (3.8), we have $\Delta = 16H^4\omega^4$. System (3.1) no longer splits into two independent systems. The characteristic equation of the precession system obtained from (3.1) may be written in the form

$$\begin{gathered}(1 + \varepsilon_1\varepsilon_2)\lambda^4 + (m_1 + m_2)\lambda^3 + (2\omega^2 + m_1m_2)\lambda^2 + \omega^2(m_1 + m_2)\lambda + \omega^4 = 0 \\ \varepsilon_i = b_i/(2H), \quad m_i = s_i/(2H); \quad i = 1, 2\end{gathered} \tag{3.11}$$

Since the coefficients of this equation are positive, the single Hurwitz condition for the equation may be reduced to the form

$$\omega^2(s_1 + s_2)^2(s_1s_2 - b_1b_2\omega^2) > 0 \tag{3.12}$$

Condition (3.12) will hold if

$$\omega \neq 0, \quad s_1s_2 > b_1b_2\omega^2 \tag{3.13}$$

When conditions (3.13) are satisfied, we obtain asymptotic stability in the precession equations. If we take $s_1 = s_2 = s$, $b_1 = b_2 = b$, the second of conditions (3.13) becomes $s > b\omega$, in agreement with inequality (3.7), in which, with reference to the case $c = 0$, we must also assume that $\omega \neq 0$.

If the device is capable of reacting to the Earth's rotation, we must set $\omega = U\sin\varphi$ in the case of a base which is stationary with respect to the Earth. Then condition (3.12) is satisfied only if the second of conditions (3.13) is satisfied, except at the equator, when $\varphi = 0$. Otherwise, one can apply a forced rotation of the base at a given angular velocity. This measure has been applied in practice, for the purpose of increasing the precision of the gyroscopic device, and is described in the technical literature [11].

If $\omega \neq 0$, the last two equations of system (3.1) contain terms $2H\omega\alpha$ and $2H\omega\beta$ with the factor $2H\omega$, which in turn appears in the structure of the matrix Π of conservative forces in (3.2).

Therefore, rotation of the base at an angular velocity ω promotes the appearance of conservative properties in the system as a whole. Under certain conditions, this will extend the region of stability of the device considered.

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